Exponential Asymptotics & Free-Surface Flows

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ABSTRACT

When traditional linearised theory is used to study free-surface flows past a surface-piercing object or over an obstruction in a stream, the geometry of the object is usually lost, having been assumed small in one or several of its dimensions. In order to preserve the nonlinear nature of the geometry, asymptotic expansions in the low-Froude or low-Bond limits can be derived, but here, the solution invariably predict a waveless free-surface at every order. This is because the waves are in fact, exponentially small, and thus *beyond-all-orders* of regular asymptotics; their formation is a consequence of the divergence of the asymptotic series and the associated Stokes Phenomenon.

In this thesis, we will apply exponential asymptotics to the study of two new problems involving nonlinear geometries. In the first, we examine the case of free-surface flow over a step including the effects of both gravity and surface tension. Here, we shall see that the availability of multiple singularities in the geometry, coupled with the interplay of gravitational and cohesive effects, leads to the discovery of a remarkable new set of solutions.

In the second problem, we study the waves produced by bluff-bodied ships in low-Froude flows. We will derive the analytical form of the exponentially small waves for a wide range of hull geometries, including single-cornered and multi-cornered ships, and then provide comparisons with numerical computations. A particularly significant result is our confirmation of the thirty-year old conjecture by Vanden-Broeck & Tuck (1977) regarding the impossibility of waveless single-cornered ships.

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PREFACE

> *The construction and acceptance of the theory of divergent series is another striking example of the way in which mathematics has grown…it demonstrates how far mathematicians have come to recognize that mathematics is man-made. The definitions of summability are not the natural notion of continually adding more and more terms...they are artificial.*
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> —MORRIS KLINE, (1972)

Today, some of the most exciting scientific advances are made at the boundaries between fundamentally different theories of the universe—from classical mechanics to quantum mechanics, geometrical optics to wave optics, viscous flow to inviscid flow, and so on. The difficulty, however, is that the transition from one physical theory to the next occurs in a *singular* limit; as such, we must take great care in recognising the subtle and unexpected effects that often occurs in their study. This thesis, broadly speaking, is about these very effects.

Now as mathematicians, we know that the study of such singular problems typically leads to the solution, S, being represented as an asymptotic expansion,

$$S = \sum_{n=0}^{\infty} \epsilon^n S_n,$$

which depends on the limit of the small parameter, ϵ, tending to zero. But the limit that ϵ *tends* to zero differs qualitatively from when ϵ is *equal* to zero, and difficulties arise; the series S is divergent and consequently, by the Stokes Phenomenon, exponentially small terms can suddenly appear or disappear as the series is continued past critical lines in the complex plane. The resolution of the original problem often hinges on the effects of these tiny terms, but because such *beyond-all-orders* features are invisible to a traditional ϵ-power series, special techniques called *exponential asymptotics* (or asymptotics beyond-all-orders) must be used for their detection.

For those of you who have yet to encountered these cryptic properties of asymptotics, then the above paragraph likely resembles *hocus-pocus* moreso than rigorous mathematics! The truth is, the use of divergent expansions did indeed spark one of the great controversies of mathematics, and the concept has painstakingly traversed the great rungs of public opinion, from sour dissension (Niels Abel famously wrote that “divergent series are the invention of the devil, and it is shameful to base on them any demonstration whatsoever”) to acknowledgment and acceptance. Today, the use of divergent expansions is widespread in the applied sciences, and moreover, exponential asymptotics has emerged as a field of its own, a direct consequence of parallel developments in no less than nine interrelated fields of mathematics, physics, and engineering (Boyd, 1999).

Although the birth of these beyond-all-orders techniques may be attributed to the early investigations of Stieltjes (1886) and Stokes (1902), it was not until later applications to physically relevant problems that the theory flourished; some of these inspirational problems include Kruskal and Segur's (1991) work on the crystal growth problem, work on the viscous fingering problem in Hele-Shaw cells (*e.g.* in Combescot et al. 1986, 1988; Chapman 1999), and work on the exponentially-small waves of the fifth-order Korteweg-de Vries equation (*e.g.* in Combescot et al. 1988; Grimshaw & Joshi 1995). This thesis, however, is most directly motivated by the methods pioneered by Chapman and Vanden-Broeck (2002; 2006) for applying exponential asymptotics to the study of low-speed water waves disturbed by an obstruction.

The material in this thesis naturally splits between two problems, one about gravity-capillary theory, and the other about waveless ships. Because of the sharp distinction, every effort has been made to structure the thesis so as to allow selective reading; effectively, you may choose to begin your journey with the gravity-capillary waves of Chapter 1, or instead, you may join me in search of the fabled waveless ships of Chapter 3. The techniques that underly each work, as well as the relevant literature, are introduced separately and independently at the beginning of each of these chapters, and final remarks are made in Chapter 5.

Whichever path you choose, I daresay that together, we shall see some truly marvellous things in the world that lies *beyond-all-orders*. The study of exponential asymptotics will bring us from the real world, through the imaginary plane of existence, and back to the real; it will show us how divergent expansions—nonsensical in the conventional sense—can be refined for further information; it will reveal waves where there were none; and it will allow us to stare, unwavering and unafraid, into the abyss of the infinity, and to emerge with new insights. The problems and work that lies herein has served to enchant my mind and spirit over the last few years. I can only hope that it will do the same for you.